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On the Rate of Quantum Ergodicity on hyperbolic Surfaces and Billiards

by

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Abstract:

The rate of quantum ergodicity is studied for three strongly chaotic (Anosov) systems. The quantal eigenfunctions on a compact Riemannian surface of genus $g = 2$ and of two triangular billiards on a surface of constant negative curvature are investigated. One of the triangular billiards belongs to the class of arithmetic systems. There are no peculiarities observed in the arithmetic system concerning the rate of quantum ergodicity. This contrasts to the peculiar behaviour with respect to the statistical properties of the quantal levels. It is demonstrated that the rate of quantum ergodicity in the three considered systems fits well with the known upper and lower bounds. Furthermore, Sarnak's conjecture about quantum unique ergodicity for hyperbolic surfaces is confirmed numerically in these three systems.

Note:

The postscript file of this paper containing all figures is available at:

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I Introduction

In classical mechanics the behaviour of systems can be classified in the range from integrable towards strongly chaotic. Thereby one of the lowest suppositions of chaos is given by ergodicity being followed by the mixing property and finally by the Anosov property. Thus as a first step towards chaos it suggests itself to study the quantum mechanical analogue of the classical ergodicity theorem (see e.g. [1]).

Here, we restrict us to bounded Hamiltonian systems having a purely discrete quantal eigenvalue spectrum $\{E_n\}$, $n \in \mathbb{N}$, with corresponding eigenstates Ψ_n which are assumed to be orthonormalized with respect to the L_2 norm. Especially, we are concerned with the eigenstates of the Laplace-Beltrami operator Δ on compact Riemannian manifolds $\mathcal{M}$ with constant negative curvature, i. e.

$$- \Delta\Psi_n \;=\; E_n \, \Psi_n \quad . \tag{1}$$

In quantum mechanics the following measure was introduced for compact Riemannian manifolds $\mathcal{M}$ [2, 3]

$$S_k(E;A) \;:=\; \frac{1}{N(E)} \sum_{E_n \leq E} \left| \, (\, A\Psi_n, \Psi_n) - \overline{\sigma}_A \, \right|^k \quad , \quad k > 0 \quad , \tag{2}$$

where $N(E) := \#\{E_n | E_n \leq E\}$ is the staircase function counting the number of the quantal levels E_n below the energy E. The operator A is a pseudo-differential operator of 0-th order, i. e. roughly speaking, it is an operator containing no derivatives. For an introduction to the theory of pseudo-differential operators, see [4, 5]. The average $\overline{\sigma}_A$ is defined as

$$\overline{\sigma}_A := \frac{1}{\mathrm{vol}(S^\star\mathcal{M})} \int_{S^\star\mathcal{M}} \sigma_A \, d\tau \quad , \tag{3}$$

where $S^\star\mathcal{M}$ is the unit cotangent bundle, i. e. the energy shell in phase space. The Liouville measure on $S^\star\mathcal{M}$ is denoted by $d\tau$, and σ_A is the principal symbol of A, which can be considered as the classical analogue of A. If the flow on $S^\star\mathcal{M}$ is ergodic it is proven [6, 7, 8, 9] that

$$S_k(E;A) \;\to\; 0 \quad \text{for} \quad E \to \infty \quad . \tag{4}$$

Thus the vanishing of (2) in the limit $E \to \infty$ can be considered as a quantum analogue of ergodicity.

The definition of *quantum ergodicity* as (4) is equivalent to another formulation. For bounded sequences $\{a_n\}$, $a_n \in \mathbb{C}$, one has the following lemma [10]. The condition

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leq N} | \, a_n \, |^p \;=\; 0 \quad \text{for any} \quad p > 0$$

is equivalent to

$$\lim_{j\to\infty} a_{n_j} \;=\; 0 \quad ,$$

where $\{n_j\}$ is a subsequence of $\mathbb{N}$ of density 1. A subsequence n_j, $n_1 < n_2 < \ldots$, is defined to be of density d $(0 \le d \le 1)$ if

$$\lim_{N\to\infty} \frac{\#\{n_j | n_j \le N\}}{\#\{n | n \le N\}} \; = \; d \quad .$$

Thus the condition (4) of quantum ergodicity reads as

$$\big(A\Psi_{n_j}, \Psi_{n_j} \big) \; \to \; \overline{\sigma}_A \quad \text{for} \quad E_{n_j} \; \to \; \infty \tag{5}$$

for a subsequence $\{n_j\}$ of density one. The kind of the decay of (2) for $E \to \infty$ and (5) is coined as the *rate* of quantum ergodicity. Quantum *unique* ergodicity happens to be if the condition (5) holds for the complete sequence $\{n\}$

$$(A\Psi_n, \Psi_n) \; \to \; \overline{\sigma}_A \quad \text{for} \quad E_n \; \to \; \infty \quad . \tag{6}$$

Attention has to be paid for interpreting the condition (4) as a measure for quantum ergodicity, since one cannot conclude from (4) that the classical system is ergodic. An instructive example is the sphere S^2, which can be considered as the counterpart of the surfaces studied in this paper since the sphere S^2 has constant positive curvature. The usual basis of spherical harmonics $\{Y_l^m\}$ is indeed not quantum ergodic as becomes clear by considering subsequences of a fixed rational m/l. However, for almost all orthogonal bases the eigenfunctions of the sphere S^2 are quantum ergodic [11]. An example of a subsequence of density zero violating (5) is provided by the subsequence $\{Y_l^l\}$ which is very strongly localized at the equator. If one considers, however, the subsequence $\{Y_l^m\}$ with $m/l \ge c$, $0 < c < 1$, one obtains a subsequence of density $\frac{1-c}{2} > 0$ which is concentrated in the neighbourhood of the equator given by $\sin\theta \ge c$ [12].

In the case of compact Riemannian manifolds of negative curvature the best known upper bound of $S_k(E;A)$ is given by [2]

$$S_k(E;A) \; = \; O\left((\log E)^{-\frac{k}{2}} \right) \quad . \tag{7}$$

By considering the sum over eigenstates in eq.(2) one avoids possible sparse subsequences $\{n_j\}$ of zero density, for which

$$\left| \, \big(A\Psi_{n_j}, \Psi_{n_j} \big) - \overline{\sigma}_A \, \right| \; = \; \Omega(1) \quad ,$$

which might correspond to eigenstates scarred by short classical periodic orbits. However, this does not occur for the hyperbolic octagons studied in [13]. Here Ω is the Hardy-Littlewood-lower-bound symbol being defined as

$$f(x) = \Omega(g(x)) \quad \text{for} \quad x \to \infty \quad \text{if} \quad f(x) \ne o(g(x)) \quad .$$

If the Liouville average of the sub-principal symbol $\sigma_{\text{sub}}(A\,)$ does not vanish, then a lower bound is given by [3]

$$| \, (A\Psi_n, \Psi_n) - \overline{\sigma}_A \, | \; = \; \Omega\left(E_n^{-\frac{1}{2}} \right) \quad . \tag{8}$$

Addressing the question of quantum ergodicity in individual eigenstates, a conjecture is put forward by Sarnak [14] which states that for hyperbolic surfaces in which the curvature is negative

$$(A\Psi_n, \Psi_n) - \overline{\sigma}_A \; = \; O\left(E_n^{-\frac{1}{4}+\varepsilon} \right) \qquad \text{for all} \quad \varepsilon > 0 \quad . \tag{9}$$

The main topic in this paper deals with the rate of the decay of $S_1(E;A)$. An extended analysis including higher $S_k(E;A)$, $k \geq 1$, and other statistics of the wave functions of the hyperbolic billiards considered in this paper can be found in [15]. A detailed analysis concerning the behaviour of $S_1(E;A)$ of Euclidean billiards, i. e. the stadium billiard, the cardioid billiard and the so-called cosine billiard, can be found in [16].

II The chaotic systems

Surfaces of constant negative Gaussian curvature constitute a simple model for chaos since the classical dynamics on these surfaces possess all of the characteristic properties of instability: ergodicity, mixing and the Anosov property. As a model for this surface we use the Poincaré disc $\mathcal{D}$ which consists of the interior of the unit circle in the complex z–plane ($z = x_1 + ix_2$) endowed with the hyperbolic metric

$$g_{ij} \; = \; \frac{4}{(1 - x_1^2 - x_2^2)^2}\delta_{ij} \quad , \quad i,j = 1,2 \tag{10}$$

corresponding to constant negative Gaussian curvature $K = -1$. This fixes the length scale. The systems which are studied in this paper are defined on this hyperbolic surface.

The classical motion (geodesic flow) is determined by the Hamiltonian $H = \frac{1}{2m} p_i g^{ij} p_j$, $p_i = m\, g_{ij}\, dx^j/dt$. The geodesics are circles intersecting the boundary of the Poincaré disc $\mathcal{D}$ perpendicularly.

The quantum mechanical system is governed by the Schrödinger equation

$$- \Delta\Psi_n(z) = E_n\Psi_n(z) \quad , \quad \Delta = \frac{1}{4}(1 - x_1^2 - x_2^2)^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \quad , \tag{11}$$

where Δ denotes the non–Euclidean Laplacian corresponding to the hyperbolic metric (10). Here and in the following units $\hbar = 2m = 1$ are used. The wave functions $\Psi_n(z)$ have to obey appropriate boundary conditions, see below.

Since we are interested in possible differences in the rate of quantum ergodicity between non-arithmetic and arithmetic systems, we choose the asymmetric octagon shown in figure 1 as the non-arithmetic system, and the so-called regular octagon for the arithmetic system. Both are Riemannian surfaces of genus two. (For more details about these Riemannian surfaces, see e. g. [17, 18].) However, both systems decompose into irreducible symmetry representations

such that we consider only one symmetry class. In the case of the non-arithmetic octagon which only possesses one involution symmetry the positive symmetry class is chosen, i. e. $\Psi(-z) = +\Psi(z)$. The regular octagon which has the highest symmetry possible for a surface of genus 2, decomposes into 96 symmetry classes from which the one is selected which corresponds to a hyperbolic triangle with angles $\alpha = \pi/3$, $\beta = \pi/8$ and $\gamma = \pi/2$. The sides a, b and c being opposite to their corresponding angles are endowed with the boundary conditions Dirichlet, Neumann and Neumann, respectively. Since these two desymmetrized systems differ in some aspects, we incorporate in our study a further triangular billiard which differs only in the imposed boundary conditions from the former. For the sides a, b and c the boundary conditions Neumann, Neumann and Dirichlet are chosen, respectively, which do not provide a subspectrum of the regular octagon. The main point is that this combination of boundary conditions leads to a non-arithmetic system. For a detailed investigation of these two hyperbolic triangular billiards with respect to the quantal level statistics, see [19].

Thus our study is based on one parity class of a non-arithmetic Riemannian surface and on one hyperbolic triangle endowed with two different boundary conditions from which only one is arithmetic. In the following the non-arithmetic billiard is called billiard $\mathcal{A}$ and the other billiard $\mathcal{B}$, see figure 1. Using the boundary element method [20] the first 3000 wave functions of the positive symmetry class of the asymmetric octagon are computed. For billiard $\mathcal{A}$ and $\mathcal{B}$ the first 2092 and 2099 wave functions are determined, respectively. The following statistical analysis is based on this large set of wave functions.

III On the behaviour of $S_1(E; A)$ in the configuration space

The pseudo-differential operator of 0-th order A is chosen to be the multiplication operator with the characteristic function of a subset $\mathcal{X} \subset \mathcal{M}$ on the Poincaré disc $\mathcal{D}$, i. e. for $z \in \mathcal{X}$ it is one and zero otherwise. Thus A does not depend on the momentum p. Then the scalar product in (2) reduces to

$$(A\Psi_n, \Psi_n) \;=\; \int_{\mathcal{X}} d\mu(z) \, |\Psi_n(z)|^2 \quad , \tag{12}$$

i. e. the probability to find the particle in $\mathcal{X}$. The invariant volume element on $\mathcal{D}$ is designated as $d\mu(z)$. With this choice of A the average $\overline{\sigma}_A$ is given by

$$\overline{\sigma}_A = \frac{\text{Area}(\mathcal{X})}{\text{Area}(\mathcal{M})} \quad ,$$

where Area($\mathcal{X}$) is the area of the domain $\mathcal{X}$ and Area($\mathcal{M}$) is the area of the desymmetrized domain. The average $\overline{\sigma}_A$ is exactly what is classically expected for the mean value of the classical analogue of the observable A for an ergodic system. The domain $\mathcal{X}$ is chosen to be a circular domain. For the numerical integration we choose a disc with radius R having its center at $z = 0$, i. e. at the origin in the Poincaré disc. The disc is then transformed by a linear

fractional transformation $z \to (\alpha z + \beta)/(\beta^\star z + \alpha^\star)$, $\alpha, \beta \in \mathbb{C}$. This maps the circle centered at $z = 0$ conformally to the domain $\mathcal{X}$ inside the billiards for suitably chosen α and β. An example of such a domain $\mathcal{X}$ characterizing the operator A is shown in figure 1 for the asymmetric octagon as well as for the triangular billiards. These two operators are chosen for the examples shown in figures 2, 3, 5, 6 and 11.

In order to understand $S_1(E; A)$ let us consider the summands in (2) in more detail and define

$$\tilde{a}_n \ := \ (\, A\Psi_n, \Psi_n) - \overline{\sigma}_A \quad \text{and} \quad a_n \ := \ |\, \tilde{a}_n \,| \quad . \tag{13}$$

Dealing numerically with finite energies we consider the limit $\varepsilon \to 0$ in Sarnak's conjecture (9) and introduce

$$\tilde{c}_n \ := \ \tilde{a}_n \, E_n^{1/4} \quad \text{and} \quad c_n \ := \ |\, \tilde{c}_n \,| \quad . \tag{14}$$

Thus the mean behaviour of a_n is given by $\overline{c}\, E_n^{-1/4}$ with $\overline{c}$ being the mean value of c_n. For the domains $\mathcal{X}$ shown in figure 1 we average the a_n's with a triangular shaped window function, denoted by $\langle \ldots \rangle$, with a base width of $\Delta n = 100$ and fitted them to $f(E) = c_f E^{-1/4}$ with c_f being a fit parameter. The result is shown in figure 2 for all three systems. One observes a good agreement in favor of Sarnak's conjecture and notes that the lower bound (8) is safely fulfilled.

The first 3000 numbers $\tilde{c}_n$, which are expected to be pseudo-random numbers with fluctuations independent of n, are shown in figure 3 in the case of the asymmetric octagon. One observes that the amplitude is of the same order over the whole range. Furthermore, no large exceptions are observed in agreement with Sarnak's conjecture. Also for billiard $\mathcal{A}$ and $\mathcal{B}$ no such exceptions are discovered.

Numerically one observes a Gaussian distribution of the $\tilde{c}_n$'s. For finite energies this distribution has, however, a non-zero mean. This is due to the imposed boundary conditions for the desymmetrized systems. In the desymmetrized asymmetric octagon the elliptic points, denoted in figure 1 as dots, play a special role. Because the hyperbolic octagons are Riemannian surfaces without boundary, there are no Neumann or Dirichlet boundary conditions which influence the eigenstates as in the case of the triangular billiards. It turns out that for the negative parity class, the eigenstates must vanish at the elliptic points, whereas in the case of positive parity, the eigenstates possess local extrema there. In the case of the regular octagon we consider as a special desymmetrized class the triangular billiard discussed in section II. Here Dirichlet and Neumann boundary conditions influence the eigenstates significantly near the boundaries. The special role of the boundary influences can be very clearly seen by considering the sum over the modulus of the eigenstates

$$W(z, E) \ := \sum_{E_n \leq E} |\, \Psi_n(z) \,|^2 \quad . \tag{15}$$

In figure 4 this quantity is shown for the positive parity class of the asymmetric octagon, the triangular billiard $\mathcal{A}$ with boundary conditions incompatible with the regular octagon and for the triangular billiard $\mathcal{B}$ derived from the regular octagon. One observes deviations from an

equidistribution enforced by the imposed boundary conditions near the elliptic points and near the boundary, respectively. At the Neumann boundary the amplitude is twice the mean value whereas at the Dirichlet boundary the amplitude vanishes, of course. For more details, see [4]. The crucial point is that these boundary effects slightly modify the local expectation value of $|\Psi_n(z)|^2$ far away from the boundary. This is the reason for the observed non-zero mean in the distribution of the $\tilde{c}_n$. Alternatively, this shift can also be taken into account by using orbit corrections to $\overline{\sigma}_A$ in equation (2) as discussed in section IV. With increasing energy these modification decrease but since one is numerically restricted to finite energies, we take the shift in the Gaussian into account. For the asymmetric octagon the cumulative distribution of $\tilde{c}_n$ is shown in figure 5 in comparison with

$$I_f(\tilde{c}\,) = \frac{1}{2}\mathrm{erfc}(-(\tilde{c}\,-\delta)/\sqrt{2}\beta) \tag{16}$$

where β is the standard deviation and δ the mean value of the distribution of $\tilde{c}_n$. A fit excluding the first 50 $\tilde{c}_n$'s yields the values $\beta = 0.0957\ldots$ and $\delta = -0.0029\ldots$ which is very small in comparison with β. A Kolmogorov-Smirnov test gives a significance level of 55%. Neglecting the shift δ leads to a significance level being almost zero. In the case of billiard $\mathcal{A}$ one obtains for the domain shown in figure 1 $\beta = 0.5154\ldots$ and $\delta = -0.0277\ldots$ again with a significance level of 55%. Billiard $\mathcal{B}$ gives $\beta = 0.4333\ldots$ and $\delta = -0.0237\ldots$ with a significance level of 94%. The Gaussian behaviour of $\tilde{c}_n$ implies for c_n the distribution density

$$P(c) \;=\; \frac{1}{\sqrt{2\pi}\beta}\left\{ e^{-\frac{(c-\delta)^2}{2\beta^2}} \;+\; e^{-\frac{(c+\delta)^2}{2\beta^2}} \right\} \quad \text{for} \quad c \geq 0 \quad . \tag{17}$$

Thus β and δ determine the mean value $\overline{c}$, i. e.

$$\overline{c} \;=\; \sqrt{\frac{2}{\pi}}\,\beta\, e^{-\frac{\delta^2}{2\beta^2}} \;+\; \delta\,\mathrm{erf}\left(\frac{\delta}{\sqrt{2}\beta}\right) \;\simeq\; \sqrt{\frac{2}{\pi}}\,\beta \quad \text{for} \quad |\,\delta\,| \ll \beta \quad . \tag{18}$$

Furthermore, let us define

$$z_n \;:=\; a_n - \overline{c}\,E_n^{-\frac{1}{4}} \tag{19}$$

which can be considered as a random variable with zero mean. The standard deviation γ of the distribution of $z_n E_n^{1/4} = c_n - \overline{c}$ is then given by

$$\gamma \;=\; \sqrt{\beta^2 + \delta^2 - \overline{c}^2} \;\simeq\; \sqrt{1 - \frac{2}{\pi}}\,\beta \quad \text{for} \quad |\,\delta\,| \ll \beta \quad . \tag{20}$$

With the definitions $\overline{a}_n := \overline{c}\,E_n^{-\frac{1}{4}}$ and (19), one has, replacing $N(E)$ in (2) by $\overline{N}(E)$,

$$S_1(E;A) \;\simeq\; \frac{1}{\overline{N}(E)}\left(\sum_{E_n<E}\overline{a}_n \;+\; \sum_{E_n<E} z_n\right) \quad . \tag{21}$$

The first sum describes the mean behaviour $\overline{S}_1(E;A)$ whereas the second sum fluctuates around zero.

The mean behaviour $\overline{S}_k(E;A)$ is determined by Weyl's law

$$\overline{N}(E) \simeq \frac{\text{Area}(\mathcal{M})}{4\pi} E \quad , \quad E \to \infty \quad , \tag{22}$$

neglecting the circumference term in the case of billiards $\mathcal{A}$ and $\mathcal{B}$, leading to

$$\overline{S}_1(E;A) = \frac{1}{\overline{N}(E)} \sum_{E_n<E} \overline{a}_n \simeq \frac{4\,\overline{c}}{3} E^{-\frac{1}{4}} \quad . \tag{23}$$

Now let us consider the deviations from this mean behaviour described by the second sum in (21). Assuming that the z_n's are independent random variables, the variance $V(E)$ of the sum $\sum_{E_n<E} z_n$ is given by $V(E) = \sum_{E_n<E} \nu_n^2$ with $\nu_n = \gamma\, E_n^{-1/4}$. The variance $V(E)$ can be estimated using Weyl's law (22)

$$V(E) = \sum_{E_n<E} \gamma^2 E_n^{-\frac{1}{2}} \simeq \frac{2\overline{N}(E)\gamma^2}{\sqrt{E}} \quad . \tag{24}$$

Thus $S_1(E;A)$ computed directly from the wave functions should lie within the range

$$S_1^{\text{model}}(E;A) := \frac{4\,\overline{c}}{3} E^{-\frac{1}{4}} \pm \frac{\sqrt{2}\gamma}{\sqrt{\overline{N}(E)}\, E^{\frac{1}{4}}} \quad , \tag{25}$$

if the deviations are compared with one standard deviation and if the above assumptions are correct. This curve is determined from the two parameter $\overline{c}$ and γ which in turn are determined by β and δ. A comparison of (25) with the directly computed $S_1(E;A)$ provides in this way also a test of the exponent $\frac{1}{4}$ in (9). Figure 6 demonstrates in the case of the asymmetric octagon that the rate of quantum ergodicity for $S_1(E;A)$ is indeed given by the power $\frac{1}{4}$. In contrast to the discussion above, the shown model curve contains the first k true a_n's, i. e.

$$\begin{aligned}
\tilde{S}_1^{\text{model}}(E;A) &= \frac{1}{\overline{N}(E)} \left(\sum_{E_n\leq E_k} a_n + \sum_{E_k<E_n\leq E} \overline{a}_n \right) \\
&= \frac{1}{\overline{N}(E)} \sum_{E_n\leq E_k} a_n + \frac{4\overline{c}}{3} \left[E^{-1/4} - E_k^{-1/4} \right] \quad .
\end{aligned} \tag{26}$$

This modification was necessary since for a given domain $\mathcal{X}$ the lowest wave functions have not enough structure within $\mathcal{X}$ in order to provide pseudo-random numbers z_n. The value of k depends on the size of the domain $\mathcal{X}$. In this case $k = 50$ was chosen but already $k = 10$ gives a reasonable approximation. Note that 3000 wave functions are included.

Alternatively, we study the rate of decline of $S_1(E;A)$ by fitting to $S_1(E;A)$ the function

$$f(E) \ := \ \alpha\, E^{-\rho} \tag{27}$$

and compare ρ with the expected $\frac{1}{4}$. For our three systems we choose a sequence of ten domains $\mathcal{X}_i$. Ten circles centered at the origin of the Poincaré disc are chosen which are mapped by the same linear fractional transformation such that the considered domains do not contain elliptic points. Within a sequence the radius of the domains are chosen such that the area of the domains increases in constant steps of 1/10-th of the area of the largest domain. The two sequences are shown in figure 7 for the asymmetric octagon and for the triangular billiards. In the case of the asymmetric octagon the selected domains $\mathcal{X}_i$ have no points which are mapped by the transformation $z \to -z$ to points within the same domain thus eliminating dependencies by the involution symmetry. In figure 8 the obtained $S_1(E;A)$ are shown together with the fit (27). The functions (27) with the fitted parameters are only shown over the interval used for the fit. The values of the fit parameter α and ρ are given in table 1. One observes for the values of ρ a good agreement with the conjectured value $\frac{1}{4}$. This result clearly shows that the upper bound (7) is valid for the energies considered here. The agreement is better in the case of the asymmetric octagon. In the case of the two triangular billiards slightly larger deviations occur. In table 2 the values of the fit parameter ρ are shown depending on the interval for which the fit is carried out. A tendency is observed that ρ approaches the conjectured value $\frac{1}{4}$ for higher energies. Using domains $\mathcal{X}_i$ which are centered at other locations in $\mathcal{M}$ leads to other values of ρ, some above $\frac{1}{4}$ and some below. It turns out that the average over these ρ's is very close to $\frac{1}{4}$ as it is demonstrated for ten domains having different centers in [15]. Thus numerical evidence is given in favor of the conjecture.

The value of α depends on the area as shown in figure 9. In order to compare the three systems we plotted α versus the index $i = 1, \ldots, 10$ of the domains $\mathcal{X}_i$. Since Area($\mathcal{X}_i$) increases linearly with i, α is also plotted versus the area. One observes a linear behaviour of α over a wide range. The asymmetric octagon shows the linear behaviour up to $i = 7$, and the curve belonging to billiard $\mathcal{B}$ flattens at $i \simeq 9$. In the limit Area($\mathcal{X}$) $\to$ Area($\mathcal{M}$) the values of a_n decrease rapidly because of the normalization of the wave functions with respect to $\mathcal{M}$. This in turn leads to a decline of α for areas $\mathcal{X}$ approaching $\mathcal{M}$. Thus one has three different regions for α, one linear region for small domains, a transition region and finally a region with decreasing α.

IV The orbit theory

In this section we consider the corrections to $\overline{\sigma}_A$ which are provided by the orbit theory. Furthermore, the question is posed for possible differences in the properties of the wave functions between arithmetic and non-arithmetic systems. This question is enforced by the fact that the properties of the energy level statistics are quite different [21, 22]. This fingerprint is caused by

asymmetric octagon			billiard $\mathcal{A}$			billiard $\mathcal{B}$		
Area	α	ρ	Area	α	ρ	Area	α	ρ
0.2691	0.036	0.251	0.0031	0.101	0.234	0.0031	0.076	0.211
0.5383	0.057	0.251	0.0063	0.122	0.209	0.0063	0.134	0.221
0.8074	0.069	0.249	0.0094	0.168	0.215	0.0094	0.151	0.212
1.0766	0.082	0.250	0.0126	0.223	0.224	0.0126	0.182	0.215
1.3457	0.096	0.257	0.0157	0.266	0.226	0.0157	0.219	0.220
1.6149	0.108	0.262	0.0189	0.309	0.229	0.0189	0.272	0.229
1.8840	0.121	0.268	0.0220	0.360	0.234	0.0220	0.304	0.232
2.1532	0.119	0.261	0.0252	0.419	0.239	0.0252	0.339	0.235
2.4223	0.115	0.254	0.0283	0.480	0.244	0.0283	0.353	0.234
2.6915	0.115	0.253	0.0315	0.550	0.249	0.0315	0.367	0.233

Table 1: The fit parameters α and ρ are shown for the three systems for the sequences of domains $\mathcal{X}_i$ shown in figure 7.

the exponentially degenerated length spectrum of the periodic orbits in the case of arithmetic systems. It is thus important to look at the description of the wave functions in terms of classical orbits.

The orbit theory derived in [23] allows to express a nearly arbitrarily weighted sum over wave functions by a corresponding sum over classical orbits. Here, in contrast to the periodic orbit theory regarding the eigenvalue spectrum, all non-periodic orbits are necessary, since it is not a trace formula. For compact Riemannian surfaces the orbit formula can be stated as

$$\sum_{n=0}^{\infty} h(p_n)\ \Psi_n^\star(z)\ \Psi_n(z') \ = \ \frac{1}{2\pi} \sum_{b\in\Gamma} \widehat{h}\left(\cosh d(z,b(z'))\right) \quad , \tag{28}$$

where $h(p)$ is an even function, which is holomorphic in the strip $|\Im p| \leq \frac{1}{2}+\varepsilon, \varepsilon > 0$ and of order $h(p) = O(p^{-2-\delta}), \delta > 0$ for $|p| \to \infty$. The sum on the right-hand-side runs over the Fuchsian group Γ which defines the tessellation of the Poincaré disc $\mathcal{D}$. Furthermore, $\widehat{h}(y)$ is the Mehler transform of $h(p)$

$$\widehat{h}(y) \ = \ \int_0^\infty dp\, p\, \tanh(\pi p)\, P_{-\frac{1}{2}+ip}(y)\, h(p) \tag{29}$$

with $P_{-\frac{1}{2}+ip}(y), y > 1$, being the Legendre function of the first kind. The length $d(z,b(z'))$ is the length of the path connecting z and z' with the topology according to $b \in \Gamma$ which is non-periodic for $z = z'$ in general. Only in the special case in which $z = z'$ lies on the periodic orbit belonging to b, the length of a periodic orbit is obtained. It is the lengths of these periodic orbits which are so peculiar degenerated in arithmetic systems. However, since for the description of the wave functions according to (28) the distribution of the lengths $d(z,b(z'))$ is important, the

billiard $\mathcal{A}$				billiard $\mathcal{B}$			
Area	ρ_1	ρ_2	ρ_3	Area	ρ_1	ρ_2	ρ_3
0.0031	0.218	0.236	0.234	0.0031	0.202	0.222	0.225
0.0063	0.177	0.206	0.235	0.0063	0.234	0.229	0.235
0.0094	0.163	0.220	0.243	0.0094	0.237	0.218	0.221
0.0126	0.160	0.229	0.259	0.0126	0.233	0.227	0.219
0.0157	0.154	0.230	0.273	0.0157	0.229	0.229	0.227
0.0189	0.150	0.237	0.267	0.0189	0.230	0.223	0.245
0.0220	0.141	0.244	0.255	0.0220	0.240	0.217	0.251
0.0252	0.146	0.252	0.254	0.0252	0.250	0.217	0.255
0.0283	0.151	0.258	0.254	0.0283	0.264	0.213	0.249
0.0315	0.150	0.259	0.260	0.0315	0.276	0.211	0.244

Table 2: The fit parameter ρ is shown for the two billiards in dependence on the fit interval: ρ_1 is fitted over $n = 200 \ldots 800$, ρ_2 over $n = 800 \ldots 1400$ and ρ_3 over $n = 1400 \ldots 2000$.

crucial question is, whether these lengths are, for general z and z', differently distributed in arithmetic and non-arithmetic systems.

IV.1 The distribution of $d(z, b(z))$

To emphasize the role of $d(z, b(z))$ we specialize (28) to a single wave function by choosing as in [23]

$$h(p) \;=\; \coth(\pi p)\, e^{-(p-p')^2/\varepsilon^2} \;-\; \coth(\pi p)\, e^{-(p+p')^2/\varepsilon^2} \quad , \quad \varepsilon > 0 \quad . \tag{30}$$

Approximating the Mehler transform for large p one gets the semiclassical expression (eq.(37) in [23])

$$\sum_{n=0}^{\infty} h(p_n)\; |\Psi_n(z)|^2 \;=\; \frac{p'\,\varepsilon}{\sqrt{4\pi}} \;+\; \sum_{b\in\Gamma'} \frac{\sqrt{p'}\,\varepsilon \sin(p'\tau_b + \frac{\pi}{4})}{\sqrt{2\pi^2 \sinh \tau_b}}\; e^{-(\tau_b \varepsilon/2)^2} \quad , \tag{31}$$

where $\tau_b := d(z, b(z))$, and Γ' is the Fuchsian group without the identity element which contribution has to be considered separately leading to the first term on the right side. The sum over Γ' is absolutely convergent for $\varepsilon > 0$. Choosing ε sufficiently small such that on the left-hand-side only the summand belonging to $p' = p_n$ contributes, allows the computation of a wave function solely from the Fuchsian group Γ. Therefore, the modulus of a wave function $|\Psi_n(z)|$ is determined solely by the lengths τ_b.

Thus to address the question whether there are differences between arithmetic and non-arithmetic systems with respect to their wave functions, one can also investigate the properties of the lengths τ_b of arithmetic and non-arithmetic systems. The simplest statistic is the nearest-neighbour length spacing distribution $P(\Delta\ell, z)$. Since the considered systems are invariant

under time reversal, all lengths occur twice in the sum over Γ which is expressed by the fact that $\tau_b = \tau_{b^{-1}}$ with $b\,b^{-1} = \mathbf{1}$. Thus in the statistic only one of the two lengths has to be included. The next step is the unfolding of the length spectrum using the asymptotic law [23]

$$N(z,z',\tau) \;\sim\; \overline{N}(z,z',\tau) \;=\; \frac{1}{4}e^{\tau} \quad , \quad \tau \to \infty \tag{32}$$

where $N(z,z',\tau)$ is the staircase function

$$N(z,z',\tau) \;:=\; \#\{b\,|\,b\in\Gamma, d(z,b(z'))\leq\tau\} \quad . \tag{33}$$

The unfolded lengths are denoted by $\ell(z,z') = \overline{N}(z,z',\tau_b)$. For the asymmetric and the regular octagon all elements $b \in \Gamma$ are computed which can be represented as a product of the generators of the group with up to 12 factors. Both groups contain only hyperbolic elements. (Elliptic and reflection elements occur later, when we consider the group for the desymmetrized systems.) It turns out that the length spacing distribution $P(\Delta\ell, z)$ is for almost all z conform to a Poisson distribution $P_{\text{Poisson}}(\Delta\ell) = e^{-\Delta\ell}$. In the case of the asymmetric octagon one finds strong deviations from the Poisson distribution at positions of the elliptic points of the desymmetrized system, which are marked in figure 1 as dots. In order to test the agreement with the Poisson distribution we apply the Kolmogorov-Smirnov test to the distributions in dependence of the position z. The results are shown in figure 10. Here the reverse of the significance level $\mathcal{P}$ is plotted, i. e. the low regions are the domains with a Poisson distributed length spacing. It is clearly seen that in the case of the asymmetric octagon only the elliptic points show strong non-Poisson behaviour whereas the regular octagon displays also deviations along symmetry lines due to the reflection elements which tessellate the regular octagon in just the domains of the triangular billiard. However, inside the triangular billiard the lengths τ_b are Poisson distributed within the same significance level as in the asymmetric octagon excluding the elliptic points.

Recalling that the main difference between arithmetic and non-arithmetic systems arises from the exponentially increasing degenerations among the periodic orbits with respect to their lengths, one now sees that such a behaviour does not occur with respect to τ_b for almost all z. Thus, no difference is expected between the wave functions of arithmetic and non-arithmetic systems. There might be, however, the possibility that the arithmetic structure of the group leads to higher correlations among the τ_b's which are independent of z and simultaneously affect the sum on the right-hand-side of eq.(31), but this seems to be unlikely.

IV.2 Corrections to $\overline{\sigma}_A$

The other important application of the orbit theory is the derivation of corrections to the leading term $\overline{\sigma}_A$ in terms of the groups Γ. Let us now consider the groups for the desymmetrized systems. In the case of the asymmetric octagon the group is extended by $S := \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ being a realization of the parity operation $z \to -z$ [24]. This group contains also elliptic elements. The

triangular billiard is described by the reflection group which is generated by the three reflection elements which realize the reflections along the three edges of the billiard. This group contains in addition to the elliptic elements also reflection elements. (For more details, see e. g. [25].) The formula eq.(28) is also valid for these groups if one takes for Γ the above groups and takes the character $\chi(b)$ of the group elements $b \in \Gamma$ on the right-hand-side of (28) into account.

To obtain the corrections to $\overline{\sigma}_A$ we choose in eq.(28)

$$h(p) \;=\; \Theta(p_{\max} - |p|\,) \quad , \tag{34}$$

where Θ is the Heaviside step function. This leads to

$$\begin{aligned} \sum_{n=0}^{N} |\,\Psi_n(z)\,|^2 \;&=\; \frac{1}{2\pi}\int_0^{p_{\max}} dp\; p \tanh(\pi\, p) \\ &\quad + \frac{1}{2\pi}\sum_{b\in\Gamma'} \chi(b) \int_0^{p_{\max}} dp\; p \tanh(\pi\, p)\, P_{-\frac{1}{2}+ip}(\cosh(\tau_b)\,) \quad , \end{aligned} \tag{35}$$

with N such that $p_N \leq p_{\max} < p_{N+1}$. The left-hand-side is now rewritten as a Riemann-Stieltjes integral

$$\sum_{n=0}^{N} |\,\Psi_n(z)\,|^2 \;=\; \int_0^{p_{\max}} |\,\Psi_n(z)\,|^2\, dN(p) \;\simeq\; \int_0^{p_{\max}} |\,\Psi_n(z)\,|^2\, d\overline{N}(p) \quad , \tag{36}$$

where in the last step the approximation

$$dN(p) \;\simeq\; d\overline{N}(p) \;=\; \left(\frac{\text{Area}(\mathcal{M})}{2\pi} p \,+\, \frac{\mathcal{L}}{4\pi}\right)\, dp$$

has been used together with Weyl's law including the circumference term. The length $\mathcal{L}$ is defined as $\mathcal{L} := \mathcal{L}^+ - \mathcal{L}^-$ whereby $\mathcal{L}^+$ and $\mathcal{L}^-$ are the lengths of the boundary on which Neumann and Dirichlet boundary conditions are imposed, respectively. The comparison of the integrand of (35) with (36) yields

$$|\,\Psi_n(z)\,|^2 \;=\; \frac{\tanh(\pi\, p_n)}{\text{Area}(\mathcal{M}) \,+\, \frac{1}{2p_n}\mathcal{L}} \left\{ 1 \,+\, \sum_{b\in\Gamma'} \chi(b)\; P_{-\frac{1}{2}+ip_n}(\cosh(\tau_b)\,) \right\} \quad . \tag{37}$$

In contrast to (31) this expression is at most conditionally convergent. Integrating equation (37) over a domain $\mathcal{X}$ leads to an orbit expression for $(A\Psi_n, \Psi_n)$ as defined in (12). The term corresponding to the identity element in eq.(37) yields $\overline{\sigma}_A$. The influence of the boundary, i. e. of the reflection elements, and of the corners, i. e. of the elliptic elements, vanishes in the semiclassical limit such that the properties of the a_n's are dominated by the contribution of the

hyperbolic elements. Thus one possible extrapolation to the semiclassical limit is provided by introducing

$$\overline{\sigma}_A^{\rm corr} \; = \; \left\{ \overline{\sigma}_A \; + \; \frac{1}{\text{Area}(\mathcal{M})} \sum_{b\in\Omega} \; \chi(b) \; \int_{\mathcal{X}} \; d\mu(z) \; P_{-\frac{1}{2}+ip_n}(\cosh(\tau_b)\,) \right\} \; \left(1 - \frac{\mathcal{L}}{2\,p_n \, \text{Area}(\mathcal{M})} \right) \tag{38}$$

The definition of $\Omega \subset \Gamma'$ is arbitrarily and depends on the effects one wishes to incorporate in $\overline{\sigma}_A^{\rm corr}$. In the case of the two triangular billiards $\mathcal{A}$ and $\mathcal{B}$ only the three reflection elements which realize the reflections at the boundary of the billiards are included in our consideration. It turns out that they cause the main contribution and that the contribution of further reflection elements as well as of elliptic elements are negligible for the domains considered here. However, choosing domains which contain elliptic points leads to important contributions from just the corresponding elliptic elements. In the case of the asymmetric octagon one has no reflection elements such that the elliptic elements have to be considered. The next class are the hyperbolic group elements whose inclusion in Ω is a matter of point of view. Since the orbits spread uniformly over the phase space in ergodic systems one suspects that they are responsible for the quantum ergodicity of the wave functions and thus should not be included in (38) since it is just this effect one wants to study.

The effect of the reflection elements is well demonstrated by the quantity

$$\kappa_n \; := \; \frac{(\, A\Psi_n, \Psi_n)}{\overline{\sigma}_A} \tag{39}$$

and its average $\langle \kappa_n \rangle$, where $\langle \ldots \rangle$ denotes the averaging with the same triangular shaped window function as above. According to (9) this tends towards one. The result is shown in figure 11 for the asymmetric octagon and for both billiards using the domains $\mathcal{X}$ shown in figure 1. As expected, the deviations from one are the larger the smaller n is, showing directly the influence of the boundary. In the case of the asymmetric octagon the deviations are negligible for $n > 700$ showing only a minute influence of the elliptic elements. The larger deviations of the two billiards are well explained by the contribution of the reflection elements. Integrating (37) over the domain $\mathcal{X}$ and taking only the reflection elements into account provides the dashed curves shown in figure 11. Not only the mean behaviour of the deviations are explained but also the long-range oscillations. Subtracting this contribution from the κ_n and averaging the result gives the dotted curves in figure 11. These fluctuate around one already for very small values of n. The remaining fluctuations are due to the contributions of other group elements not included in $\Omega \subset \Gamma'$. The order of the amplitude of these fluctuations has to be compared with the standard deviation $\Delta\kappa$ of the fluctuations of the κ_n's which are not averaged. For billiard $\mathcal{A}$ and $\mathcal{B}$ one gets $\Delta\kappa = 0.127$ and $\Delta\kappa = 0.106$, respectively, which is orders of magnitudes larger than the range shown in figure 11. Computing $c_n^{\rm corr} := |(A\Psi_n, \Psi_n) - \overline{\sigma}_A^{\rm corr}| E^{1/4}$ and applying the fit (16) as in section III gives for the $c_n^{\rm corr}$'s for billiard $\mathcal{A}$ $\beta = 0.5151\ldots$ and $\delta = -0.0127\ldots$ and for billiard $\mathcal{B}$ $\beta = 0.4333\ldots$ and $\delta = 0.0018\ldots$, respectively. Thus the width of the distribution is

unchanged but the shift δ of the Gaussian is drastically reduced. The significance levels turn out to be unaltered.

We have computed $S_1(E;A)$ with these corrected a_n's but found no significant improvement concerning the fit values of ρ. It thus seems, that for some other reasons the two billiards approach slower the suggested value $\frac{1}{4}$ than the asymmetric octagon.

V Summary and discussion

In this paper we have investigated the behaviour of the wave functions of three strongly chaotic (Anosov) systems. The rate of quantum ergodicity is studied by the limit $E \to \infty$ of $S_1(E;A)$ defined in (2) and (12) and is found to be consistent with the conjectured decline proportional to $E^{-1/4}$. The average of the individual summands a_n, occurring in the sum in (2), is shown to decline also proportional to $E^{-1/4}$. The $\tilde{c}_n := \tilde{a}_n E_n^{1/4}$, $\tilde{a}_n$ defined in eq.(13), are distributed according to a Gaussian with a non-zero mean for finite energies. Addressing the unique quantum ergodicity hypothesis we find no significant exceptional behaviour in individual $\tilde{c}_n$'s, whereby we choose domains far away from elliptic points and boundary lines of the considered systems. These regions play a special role as it is demonstrated by the sum (15) of the modulus of the wave functions as seen in figure 4. Sequences of domains $\mathcal{X}_i$ with increasing area are constructed for which the computed $S_1(E;A)$ are compared with $f(E) = \alpha E^{-\rho}$. The values of the fit parameter ρ are consistent with the expected value $\frac{1}{4}$. The dependence of α on Area$(\mathcal{X}_i)$ is linear for Area$(\mathcal{X}_i) \ll$ Area$(\mathcal{M})$ and declines thereafter towards zero for Area$(\mathcal{X}_i) \to$ Area$(\mathcal{M})$ after passing a transition region.

The above analysis shows no differences between the arithmetic and the non-arithmetic systems. The orbit theory, which allows to express the wave functions in terms of classical orbits, is applied to individual wave functions showing that the modulus of the wave functions depends only on the hyperbolic distances $\tau_b := d(z, b(z))$, see eq.(31). Since the peculiarities of the statistics of the quantal levels of arithmetic systems are traced back to the exponential degeneration of the lengths of the periodic orbits, one has to pose the analogous question with respect to the wave functions. In this case it is, however, the statistical properties of the lengths τ_b which determine possible peculiarities in arithmetic systems. The nearest-neighbour length spacing $P(\Delta\ell, z)$ of the unfolded τ_b's is Poisson distributed within the same significance level in all three systems. Only at the elliptic points of the asymmetric octagon and along the boundaries of the billiards the distribution is non-Poisson, as the Kolmogorov-Smirnov test presented in figure 10 shows. Thus no peculiarities for the wave functions should be expected in arithmetic systems in contrast to the quantal levels.

The deviations of $(A\Psi_n, \Psi_n)$ from $\overline{\sigma}_A$ due to the influences of the boundary conditions are computed in terms of the reflection elements in the case of the two billiards. The result presented in figure 11 shows that the main contribution arises from the three reflection elements realizing the reflection at the boundaries of the triangular billiards.

Therefore, our analysis gives strong numerical support for the quantum unique ergodicity hypothesis in the case of dynamical systems defined on hyperbolic surfaces independent of arithmetic or non-arithmetic properties.

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Figure captions:

Figure 1: On the left side the fundamental domain $\mathcal{M}$ of the asymmetric octagon is shown on the Poincaré disc $\mathcal{D}$. The involution symmetry is clearly visible. The chosen operator A acts non-vanishingly on the grey disc $\mathcal{X}$. The dots denote the position of the elliptic points on this surface. On the right side the triangular billiard is presented with the two boundary conditions chosen here.

Figure 2: The average $\langle a_n \rangle$ over 100 neighbouring values for the domains $\mathcal{X}$ shown in figure 1. The corresponding fits $f(E) = c_f E^{-1/4}$ are shown as dashed curves.

Figure 3: The normalized pseudo-random numbers $\tilde{c}_n$ are displayed for the asymmetric octagon.

Figure 4: $W(z, E)$ is shown for the asymmetric octagon, for billiard $\mathcal{A}$ and for billiard $\mathcal{B}$. The triangular billiards are rotated such that one looks on the Dirichlet boundary. For the asymmetric octagon we have $E = 6\,000$ and for the two billiards $E = 200\,000$.

Figure 5: The cumulative distribution $I(\tilde{c}\,)$ of the normalized numbers $\tilde{c}_n$ is shown in comparison with a cumulative Gaussian (dashed curve) for the same operator A as in figure 3.

Figure 6: $S_1(E; A)$ is shown for the asymmetric octagon as a full curve. The dashed curve is obtained by evaluating the sum with the mean values $\overline{a}_n$ depending only on $\overline{c}$. However, the first 50 correct terms a_n have been included, see eq.(26). The dotted curves correspond to one standard deviation from the mean behaviour thus demonstrating the decline with the power $\frac{1}{4}$.

Figure 7: The chosen sequences of domains $\mathcal{X}_i$ are shown for the considered systems.

Figure 8: $S_1(E; A)$ and the fit (27) are shown for the asymmetric octagon in a), for billiard $\mathcal{A}$ in b) and for billiard $\mathcal{B}$ in c). The 10 different curves belong to the 10 domains shown in figure 7.

Figure 9: The fit parameter α is plotted for the three systems in dependence on the area of the domain $\mathcal{X}_i$. The full curve corresponds to the asymmetric octagon, the dotted curve to billiard $\mathcal{A}$ and the dashed curve to billiard $\mathcal{B}$.

Figure 10: The reverse of the significance level $\mathcal{P}$ that the unfolded length spacing of $d(z, b(z))$ is distributed according to a Poisson distribution, is shown.

Figure 11: The quantity $\langle \kappa_n \rangle$ is shown for the asymmetric octagon (dashed-dotted curve) and for both billiards (full curves) for the domains $\mathcal{X}$ shown in figure 1. The lowest curve belongs to billiard $\mathcal{B}$, and the curve belonging to billiard $\mathcal{A}$ lies above it. The contribution of the identity and the three most important reflection elements according to (37) is displayed as dashed curves. Subtracting this contribution from κ_n and averaging the result yields the dotted curves.